DING PROJECTIVE COMPLEXES WITH RESPECT TO A SEMIDUALIZING MODULE†

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Abstract. Let R be a commutative ring and C a semidualizing R-module. In this article, we introduce and investigate the notion of D_C-projective complexes. We first prove that a complex X is D_C-projective if and only if each degree of X is a D_C-projective module and $\mathcal{H}\mathrm{om}(X,H)$ is exact for any C-flat complex H. As immediate consequences of this result, some properties of D_C-projective complexes are given. Secondly, we investigate a kind of stability of D_C-projective complexes by showing that an iteration of the procedure used to define the D_C-projective complexes yields exactly the D_C-projective complexes. Finally, We introduce and characterize the notion of D_C-projective dimension of complexes.

Keywords: semidualizing modules; D_C-projective modules; D_C-projective complexes; D_C-projective dimension.

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1. Introduction and Preliminaries

In recent years, Gorenstein homological algebra has been developed to an advanced level, see for example [2, 3, 10, 16] and literatures list in them. It's main idea is to replace projective (resp. injective, flat) modules by Gorenstein projective (resp. Gorenstein injective, Gorenstein flat) modules. These modules were introduced by Enochs et al. [9, 11] as generalizations and dauls of finitely generated modules of G-dimension zero over a two-sided Noetherian ring in the sense of Auslander and Bridger [1]. At the same time, These concepts have been extended in several directions. One of generalizations is Gorenstein modules with respect to a semidualizing module. As a generalization of dualizing modules and free modules of rank 1, Foxby [12], Golod [15] and Vasconcelos [20] independently initiated the study of semidualizing modules (under different names) over a commutative Noetherian ring. In

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particular, Golod [15] used these to define G-dimension with respect a semidulizing module for finitely generated modules. Motivated by Enochs and Jenda's ideas in [9, 11], Holm and Jørgensen [17] focused on Gorenstein projective (resp. Gorenstein injective, Gorenstein flat) modules with respect to a semidualizing module C over a commutative Noetherian ring, which were called C-Gorenstein projective (resp. C-Gorenstein injective, C-Gorenstein flat) modules. White [21] extended the notions of semidualizing modules and Holm and Jørgensen's C-Gorenstein projective modules to commutative non-Noetherian rings, and she called C-Gorenstein projective modules as G_C-projective modules where C is a semidualizing module. Many classical results about the Gorenstein projectivity of modules were generalized in [21]. Dually, G_C-injective modules were studied in [22]. Holm and White [18] further extended the definition of semidualizing modules to a pair of arbitrary associative rings, and many results on semidualizing modules over commutative Noetherian rings were generalized to this more general setting. In [19], the G_C-projective modules and the G_C-projective dimension of modules over general rings were investigated. In a different direction, Enochs and García Rozas [6, 7] introduced Gorenstein projective (resp. injective, flat) complexes, and proved that over Gorenstein rings, these complexes are actually the complexes of Gorenstein projective (resp. injective, flat) modules. Yang [23] further proved that the Gorenstein projective (resp. injective) versions of the above results are true over arbitrary rings, and the Gorenstein flat version holds over coherent rings. Yang and Liang [22] introduced Gorenstein projective (resp. injective) complexes with respect to a semidualizing module C over commutative rings, and proved that these complexes are actually the complexes of G_C-projective (resp. injective) modules.

On the other hand, in [4, 5], Ding et al. considered two special cases of the Gorenstein projective and Gorenstein injective modules, which they called strongly Gorenstein flat and Gorenstein FP-injective modules, respectively. These two classes of modules over coherent rings possess many nice properties analogous to Gorenstein projective and Gorenstein injective modules over Noetherian rings. For the reason that these modules were introduced and studied by Ding and his co-authors, Gillespie renamed strongly Gorenstein flat as Ding projective, and Gorenstein FP-injective as Ding injective. In [25], Zhang, Wang and Liu introduced and studied Ding projective (resp. injective) modules with respect to a semidulaizing module over commutative rings. Ding projective (resp. injective) complexes were investigated by Yang, Liu and Liang [24], among others, they proved that over any ring R, a complex X is Ding projective (resp. injective) if and only if each X^i is a Ding projective (resp. injective) module for all $i \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(X, F)$ (resp. $\mathcal{H}\mathrm{om}(J, X)$)is exact for any flat complex F (resp. any FP-injective complex J).

Motivated by the above works, in this paper, we introduce and investigate Ding projective (resp. injective) complexes with respect to a semidulaizing module. We only deal with Ding projective complexes with respect to a semidulaizing module, Ding injective version can be given dually.

Next we shall recall some notions and definitions which we need in the later sections. In order to make things less technical, throughout this article, by a ring R, we always mean a commutative ring with identity, all modules are unitary R-modules. We use $\mathrm{Ch}(R)$ to denote the category of complexes of R-modules.

1.1 A complex

$$\cdots \longrightarrow X^{n+1} \xrightarrow{\delta^{n+1}} X^n \xrightarrow{\delta^n} X^{n-1} \longrightarrow \cdots$$

will be denoted by (X, δ) or simply X. The nth cycle (resp. boundary, homology) of X is denoted by $Z_n(X)$ (resp. $B_n(X)$, $H_n(X)$). Given an R-module M, we will denote by $\overline{M}$ the complex

$$\cdots \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \longrightarrow \cdots$$

with M in the 1 and 0th degrees. Given an $X \in \mathrm{Ch}(R)$ and an integer m, $X[m]$ denotes the complex such that $X[m]^n = X^{n-m}$ and whose boundary operators are $(-1)^m \delta^{n-m}$. Given $X, Y \in \mathrm{Ch}(R)$, We use $\mathrm{Hom}(X, Y)$ to present the group of all morphisms from X to Y, and $\mathrm{Ext}^i(X, Y)$ denote the right-derived functors of Hom. We let $\mathcal{H}\mathrm{om}(X, Y)$ denote the complex with

$$\mathcal{H}\mathrm{om}(X, Y)^n = \prod_{t \in \mathbb{Z}} \mathrm{Hom}(X^t, Y^{n+t}),$$

and with differential given by

$$\delta^n \left((f^t)_{t\in\mathbb{Z}}\right) = \left(\delta_Y^{n+t} f^t - (-1)^n f^{t-1} \delta_X^t\right)_{t\in\mathbb{Z}}.$$

1.2 ([21, 1.8]) An R-module C is called semidualizing if

(1) C admits a degreewise finite projective resolution.

(2) the natural homothety map $\chi_C^R : R \longrightarrow \mathrm{Hom}_R(C, C)$ is an isomorphism.

(3) $\mathrm{Ext}_R^{\geq 1}(C, C)$=0.

From now on, C is a fixed semidualizing R-module.

1.3 (see [21, 18]) The classes of C-projective and C-flat modules are defined as

$$\mathcal{P}_C(R) = \{C \otimes P | P \text{ is a projective module}\},$$
$$\mathcal{F}_C(R) = \{C \otimes F | F \text{ is a flat module}\}.$$

When $C = R$, we omit the subscript and recover the classes of projective and flat R-modules.

Lemma 1.1. *([18, Proposition 5.2]) Let* $0 \longrightarrow W' \longrightarrow W \longrightarrow W'' \longrightarrow 0$ *be a short exact sequence of R-modules. If* $W', W'' \in \mathcal{P}_C(R)$ *(resp.* $\mathcal{F}_C(R)$*), then* $W \in \mathcal{P}_C(R)$ *(resp.* $\mathcal{F}_C(R)$*).*

1.4 ([13]) Let $\mathcal{X}$ be a class of R-modules. A complex X is called an $\mathcal{X}$-complex if X is exact and $Z_i(X) \in \mathcal{X}$ for all $i \in \mathbb{Z}$. We let $\widetilde{\mathcal{X}}$ denote the class of $\mathcal{X}$-complexes. A complex X is called projective (resp. flat, C-pojective, C-flat), if X is a $\mathcal{P}(R)$ (resp. $\mathcal{F}(R)$, $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$)-complex.

1.5 Let $\mathcal{A}$ be an Abelian category and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. Recall that an exact sequence $\mathbf{L}$ in $\mathcal{A}$ is $\mathrm{Hom}(-,\mathcal{B})$-exact if the sequence $\mathrm{Hom}(\mathbf{L}, B)$ is exact for any $B \in \mathcal{B}$.

1.6 ([25, Definition 1.1]) An R-module M is called D_C-projective if there exists a $\mathrm{Hom}(-,\mathcal{F}_C(R))$-exact exact sequence

$$\cdots \longrightarrow X^1 \xrightarrow{\delta^1} X^0 \xrightarrow{\delta^0} X^{-1} \xrightarrow{\delta^{-1}} X^{-2} \xrightarrow{\delta^{-2}} \cdots,$$

of modules with $X^i \in \mathcal{P}(R)$ for all $i \geq 0$ and $X^i \in \mathcal{P}_C(R)$ for all $i < 0$ such that $M \cong \mathrm{Im}\delta^0$.

The class of D_C-projective R-modules denoted by $\mathcal{D}_C\mathcal{P}(R)$.

2. D_C-projective complexes

In this Section, we introduce and study D_C-projective complexes.

Definition 2.1. *A complex X is called Ding projective with respective to a semidulizing module C, simply D_C-projective, if there exists a* $\mathrm{Hom}(-,\widetilde{\mathcal{F}_C(R)})$*-exact exact sequence of complexes*

$$\cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Q_{-1} \xrightarrow{f_{-1}} Q_{-2} \xrightarrow{f_{-2}} \cdots$$

with all $P_i \in \widetilde{\mathcal{P}(R)}$ *and all* $Q_i \in \widetilde{\mathcal{P}_C(R)}$ *such that* $X \cong \mathrm{Im} f_0$.

By the definition of D_C-projective complexes, we have

Lemma 2.2. *If X is a D_C-projective complex, then* $\mathrm{Ext}^{\geq 1}(X, H)$*=0 for any* $H \in \widetilde{\mathcal{F}_C(R)}$.

Lemma 2.3. *Let* $X \in \mathrm{Ch}(R)$*. If each* $X^i \in \mathcal{D}_C\mathcal{P}(R)$ *for all* $i \in \mathbb{Z}$*, then for any* $H \in \widetilde{\mathcal{F}_C(R)}$, $\mathcal{H}\mathrm{om}(X, H)$ *if and only if* $\mathrm{Ext}^1(X, H)$*=0.*

Proof. It follows from [13, Lemma 2.1]. □

Proposition 2.4. *Let $X \in \mathrm{Ch}(R)$. If X is bounded right and each $X^i \in \mathcal{D}_C\mathcal{P}(R)$ for all $i \in \mathbb{Z}$, then $\mathrm{Ext}^1(X,H)$=0 for any $H \in \widetilde{\mathcal{F}_C(R)}$.*

Proof. Assume that $0 \longrightarrow H \xrightarrow{\mu} G \xrightarrow{\nu} X \longrightarrow 0$ is a short exact sequence in $\mathrm{Ch}(R)$ with $H \in \widetilde{\mathcal{F}_C(R)}$. It suffices to prove that this exact sequence is split.

Without loss of generality, we set

$$X = \ \cdots \longrightarrow X^2 \xrightarrow{\delta_X^2} X^1 \xrightarrow{\delta_X^1} X^0 \xrightarrow{\delta_X^0} 0 \longrightarrow 0 \longrightarrow \cdots.$$

Consider the following commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & H^2 \arrow[r, "\mu^2"] \arrow[d, "\delta_H^2"'] & G^2 \arrow[r, "\nu^2"] \arrow[d, "\delta_G^2"'] & X^2 \arrow[r] \arrow[d, "\delta_X^2"'] & 0 \\
0 \arrow[r] & H^1 \arrow[r, "\mu^1"] \arrow[d, "\delta_H^1"'] & G^1 \arrow[r, "\nu^1"] \arrow[d, "\delta_G^1"'] & X^1 \arrow[r] \arrow[d, "\delta_X^1"'] & 0 \\
0 \arrow[r] & H^0 \arrow[r, "\mu^0"] \arrow[d, "\delta_H^0"'] & G^0 \arrow[r, "\nu^0"] \arrow[d, "\delta_G^0"'] & X^0 \arrow[r] \arrow[d, "\delta_X^0"'] & 0 \\
0 \arrow[r] & H^{-1} \arrow[r, "\mu^{-1}"] \arrow[d] & G^{-1} \arrow[r, "\nu^{-1}"] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since $H \in \widetilde{\mathcal{F}_C(R)}$, X^i is D_C-projective, by Lemma 1.1 and [25, Proposition 1.4], $\mathrm{Ext}^1(X^i,H^i)$=0 for all $i \in \mathbb{Z}$. Thus there exists a $\lambda^i : G^i \longrightarrow H^i$ such that $\lambda^i\mu^i = 1_{H^i}$ for all $i \in \mathbb{Z}$. Obviously, $\lambda^i = (\mu^i)^{-1}$ is an isomorphism for all $i < 0$. Hence $\delta_H^i\lambda^i = \lambda^{i-1}\delta_G^i$ for all $i < 0$.

Note that $\delta_H^{-1}(\lambda^{-1}\delta_G^0 - \delta_H^0\lambda^0) = 0$, so $\mathrm{Im}(\lambda^{-1}\delta_G^0 - \delta_H^0\lambda^0) \subseteq \mathrm{Ker}\delta_H^{-1} = \mathrm{Im}\delta_H^0$. This implies that $\lambda^{-1}\delta_G^0 - \delta_H^0\lambda^0 \in \mathrm{Hom}(G^0, \mathrm{Im}\delta_H^0)$. On the other hand, from $(\lambda^{-1}\delta_G^0 - \delta_H^0\lambda^0)\mu^0 = 0$ it follows that $\mathrm{Ker}\nu^0 = \mathrm{Im}\mu^0 \subseteq \mathrm{Ker}(\lambda^{-1}\delta_G^0 - \delta_H^0\lambda^0)$. Thus by the Factor Lemma, there exists $\sigma^0 \in \mathrm{Hom}(X^0, \mathrm{Im}\delta_H^0)$ such that $\lambda^{-1}\delta_G^0 - \delta_H^0\lambda^0 = \sigma^0\nu^0$. Since $X^0 \in \mathcal{D}_C\mathcal{P}(R)$ and $\mathrm{Im}\delta_H^1 \in \mathcal{F}_C(R)$, $\mathrm{Ext}^1(X^0, \mathrm{Im}\delta_H^1) = 0$ by [25, Proposition 1.4]. So the sequence

$$0 \longrightarrow \mathrm{Hom}(X^0, \mathrm{Im}\delta_H^1) \longrightarrow \mathrm{Hom}(X^0, H^0) \longrightarrow \mathrm{Hom}(X^0, \mathrm{Im}\delta_H^0) \longrightarrow 0$$

is exact. Hence there exists a $\tau^0 \in \mathrm{Hom}(X^0, H^0)$ such that $\delta_H^0\tau^0 = \sigma^0$. Take $\omega^0 = \tau^0\nu^0 + \lambda^0$. Then $\omega^0 \in \mathrm{Hom}(G^0, H^0)$ and

$$\delta_H^0\omega^0 = \delta_H^0(\tau^0\nu^0 + \lambda^0) = \sigma^0\nu^0 + \delta_H^0\lambda^0 = \lambda^{-1}\delta_G^0, \ \omega^0\mu^0 = \tau^0\nu^0\mu^0 + \lambda^0\mu^0 = 1_{H^0}.$$

Since $\delta_H^0(\omega^0\delta_G^1 - \delta_H^1\lambda^1) = 0$, $\mathrm{Im}(\omega^0\delta_G^1 - \delta_H^1\lambda^1) \subseteq \mathrm{Ker}\delta_H^0 = \mathrm{Im}\delta_H^1$. This implies that $\omega^0\delta_G^1 - \delta_H^1\lambda^1 \in \mathrm{Hom}(G^1, \mathrm{Im}\delta_H^1)$. On the other hand, since $(\omega^0\delta_G^1 - \delta_H^1\lambda^1)\mu^1 = 0$, one gets $\mathrm{Ker}\nu^1 = \mathrm{Im}\mu^1 \subseteq \mathrm{Ker}(\omega^0\delta_G^1 - \delta_H^1\lambda^1)$. By the Factor Lemma, there exists a $\sigma^1 \in \mathrm{Hom}(X^1, \mathrm{Im}\delta_H^1)$ such that $\omega^0\delta_G^1 - \delta_H^1\lambda^1 = \sigma^1\nu^1$. Since $\mathrm{Ext}^1(X^1, \mathrm{Im}\delta_H^2) = 0$, we have the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(X^1, \mathrm{Im}\delta_H^2) \longrightarrow \mathrm{Hom}(X^1, H^1) \longrightarrow \mathrm{Hom}(X^1, \mathrm{Im}\delta_H^1) \longrightarrow 0.$$

Thus there exists $\tau^1 \in \mathrm{Hom}(X^1, H^1)$ such that $\delta_H^1\tau^1 = \sigma^1$. Put $\omega^1 = \tau^1\nu^1 + \lambda^1$. Then $\omega^1 \in \mathrm{Hom}(G^1, H^1)$ and

$$\delta_H^1\omega^1 = \delta_H^1(\tau^1\nu^1 + \lambda^1) = \sigma^1\nu^1 + \delta_H^1\lambda^1 = \omega^0\delta_G^1,\ \omega^1\mu^1 = \tau^1\nu^1\mu^1 + \lambda^1\mu^1 = 1_{H^1}.$$

Continue this process, we can get $\omega^i \in \mathrm{Hom}(G^i, H^i)$ with $\delta_H^i\omega^i = \omega^{i-1}\delta_G^i$ and $\omega^i\mu^i = 1_{H^i}$ for $i = 2, 3, \ldots$. Take $\omega^i = \lambda^i$ when $i < 0$. Then $\omega : G \longrightarrow H$ is a morphism with $\omega\mu = 1_H$. Therefore the sequence

$$0 \longrightarrow H \xrightarrow{\mu} G \xrightarrow{\nu} X \longrightarrow 0$$

is exact, and so $\mathrm{Ext}^1(X, H)$=0. □

Let λ be an ordinal number, $(X_\alpha)_{\alpha<\lambda}$ a family subcomplexes of a complex X. Recall that the family $(X_\alpha)_{\alpha<\lambda}$ is a continuous chain of subcomplexes [8, Definition 2.8] if $X_\alpha \subseteq X_\beta$ whenever $\alpha \leq \beta < \lambda$ and if $X_\beta = \bigcup_{\alpha<\beta} X_\alpha$ whenever $\beta < \lambda$ is a limit ordinal.

Proposition 2.5. *Let $X \in \mathrm{Ch}(R)$. If X is an exact complex and each $Z_i(X) \in \mathcal{D}_C\mathcal{P}(R)$ for all $i \in \mathbb{Z}$, then $\mathrm{Ext}^1(X, H)$=0 for any $H \in \widetilde{\mathcal{F}_C(R)}$.*

Proof. Assume that

$$X = \cdots \longrightarrow X^2 \longrightarrow X^1 \longrightarrow X^0 \longrightarrow X^{-1} \longrightarrow X^{-2} \longrightarrow \cdots$$

is an exact complex and each $Z_i(X) \in \mathcal{D}_C\mathcal{P}(R)$ for all $i \in \mathbb{Z}$. Then by [25, Theorem 1.12], each $X^i \in \mathcal{D}_C\mathcal{P}(R)$ for all $i \in \mathbb{Z}$. For any $m \geq 0$, let

$$X_m = \cdots \longrightarrow X^0 \longrightarrow X^{-1} \longrightarrow \cdots \longrightarrow X^{-m} \longrightarrow Z_{-m-1}(X) \longrightarrow 0.$$

Then $(X_m)_{m\geq 0}$ is a continuous chain of subcomplexes of X and $X = \bigcup_{m\geq 0} X_m$. Since $X_{m+1}/X_m \cong \overline{Z_{-m-1}(X)}[-m-2]$ for $m = 0, 1, \ldots$, by Proposition 2.4, $\mathrm{Ext}^1(X_0, H) = 0$, $\mathrm{Ext}^1(X_{m+1}/X_m, H) = 0$ for any $m = 0, 1, \ldots$ and any $H \in \widetilde{\mathcal{F}_C(R)}$. Therefore $\mathrm{Ext}^1(X, H) = 0$ for any $H \in \widetilde{\mathcal{F}_C(R)}$ by [8, Theorem 2.9]. □

Corollary 2.6. *If $X \in \widetilde{\mathcal{P}_C(R)}$, then $\mathcal{H}\mathrm{om}(X, H)$ is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$.*

Proof. Since $X \in \widetilde{\mathcal{P}_C(R)}$, X is an exact complex, and each $Z_i(X) \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$ by [25, Proposition 1.8]. Thus $\mathrm{Ext}^1(X, H)$=0 for any $H \in \widetilde{\mathcal{F}_C(R)}$ by Proposition 2.5. On the other hand, each $X^i \in \mathcal{D}_C\mathcal{P}(R)$ by [25, Theorem 1.12]. Thus $\mathcal{H}\mathrm{om}(X, H)$ is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$ by Lemma 2.3. □

Lemma 2.7. *Let* $0 \longrightarrow M \xrightarrow{f} L \longrightarrow N \longrightarrow 0$ *be an exact sequence of* R*-modules. If* $N \in \mathcal{D}_C\mathcal{P}(R)$, $L \in \mathcal{P}_C(R)$, *then for any* $L' \in \mathcal{P}_C(R)$ *and any* $f' \in \mathrm{Hom}(M, L')$, $\mathrm{Coker}\alpha \in \mathcal{D}_C\mathcal{P}(R)$ *where* $\alpha = (f, f') : M \longrightarrow L \oplus L'$.

Proof. Suppose that $L' \in \mathcal{P}_C(R)$, $f' \in \mathrm{Hom}(M, L')$ and $\alpha = (f, f') : M \longrightarrow L \oplus L'$. Then we have the following exact sequence

$$0 \longrightarrow M \xrightarrow{\alpha} L \oplus L' \longrightarrow \mathrm{Coker}\alpha \longrightarrow 0.$$

By the Factor Lemma, there exists an epimorphism $\mu : \mathrm{Coker}\ \alpha \longrightarrow N$ such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, equal] & L \oplus L' \arrow[r] \arrow[d, "\pi"'] & \mathrm{Coker}\alpha \arrow[r] \arrow[d, "\mu"'] & 0 \\
0 \arrow[r] & M \arrow[r, "f"] & L \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

where $\pi :\ L \oplus L' \longrightarrow L$ is the canonical projection. By the Snake Lemma, $\mathrm{Ker}\mu \cong L'$. Thus $\mathrm{Ker}\mu \in \mathcal{P}_C(R)$. So $\mathrm{Ker}\mu \in \mathcal{D}_C\mathcal{P}(R)$ by [25, Proposition 1.8]. Hence $\mathrm{Coker}\alpha \in \mathcal{D}_C\mathcal{P}(R)$ by [25, Proposition 1.10]. □

Now, we can achieve a characterization of D_C-projective complexes.

Theorem 2.8. *Let* $X \in \mathrm{Ch}(R)$. *Then* X *is a* D_C*-projective complex if and only if each* $X^i \in \mathcal{D}_C\mathcal{P}(R)$ *for all* $i \in \mathbb{Z}$ *and* $\mathcal{H}\mathrm{om}(X, H)$ *is exact for any* $H \in \widetilde{\mathcal{F}_C(R)}$.

Proof. $\Longrightarrow$) Assume that $X = \cdots \longrightarrow X^{i+1} \longrightarrow X^i \longrightarrow X^{i-1} \longrightarrow \cdots$ is a D_C-projective complex. Then there exists a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence of complexes

$$\mathbf{E} = \cdots \longrightarrow P_2 \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Q_{-1} \xrightarrow{f_{-1}} Q_{-2} \xrightarrow{f_{-2}} \cdots$$

with each $P_i \in \widetilde{\mathcal{P}(R)}$, each $Q_i \in \widetilde{\mathcal{P}_C(R)}$ such that $X \cong \mathrm{Im} f_0$. Of course for any $i \in \mathbb{Z}$, we have the following exact sequence of R-modules

$$\mathbf{E}^i = \cdots \longrightarrow P_2^i \xrightarrow{f_2^i} P_1^i \xrightarrow{f_1^i} P_0^i \xrightarrow{f_0^i} Q_{-1}^i \xrightarrow{f_{-1}^i} Q_{-2}^i \xrightarrow{f_{-2}^i} \cdots,$$

and it does have $X^i \cong \mathrm{Im} f_0^i$, where all $P_j^i \in \mathcal{P}(R)$ and all $Q_j^i \in \mathcal{P}_C(R)$ (by Lemma 1.1). Let $F \in \mathcal{F}_C(R)$. Then we have the following commutative diagram with the the top row exact

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}(Q_{-1}, \overline{F}[i]) \arrow[r] \arrow[d] & \mathrm{Hom}(P_0, \overline{F}[i]) \arrow[r] \arrow[d] & \mathrm{Hom}(P_1, \overline{F}[i]) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(Q_{-1}^i, F) \arrow[r] & \mathrm{Hom}(P_0^i, F) \arrow[r] & \mathrm{Hom}(P_1^i, F) \arrow[r] & \cdots
\end{tikzcd}$$

Now the bottom row is exact since the vertical maps are all isomorphism by [13, Lemma 3.1]. This shows that $\mathbf{E}^i$ remains exact after applying $\mathrm{Hom}(-, F)$ for any $F \in \mathcal{F}_C(R)$. Thus each $X^i \in \mathcal{D}_C\mathcal{P}(R)$ for all $i \in \mathbb{Z}$. The remainder follows from Lemma 2.2 and Lemma 2.3.

$\Longleftarrow$) Since $X^i \in \mathcal{D}_C\mathcal{P}(R)$, there exists an exact sequence of R-modules

$$0 \longrightarrow X^i \xrightarrow{f^i} Q^i \longrightarrow Y^i \longrightarrow 0$$

such that $Q^i \in \mathcal{P}_C(R)$ and $Y^i \in \mathcal{D}_C\mathcal{P}(R)$ by [25, Proposition 1.13] for all $i \in \mathbb{Z}$. For any $i \in \mathbb{Z}$ and any $(x, y) \in Q^i \oplus Q^{i-1}$, define $\delta^i(x, y) = (y, 0)$. Then

$$Q_{-1} = \cdots \longrightarrow Q^i \oplus Q^{i-1} \xrightarrow{\delta^i} Q^{i-1} \oplus Q^{i-2} \xrightarrow{\delta^{i-1}} Q^{i-2} \oplus Q^{i-3} \longrightarrow \cdots$$

is a C-projective complex. Let $i \in \mathbb{Z}$, put $\alpha_i = (f^i, f^{i-1}\sigma^i) : \ X^i \longrightarrow Q^i \oplus Q^{i-1}$. Then $\alpha = (\alpha_i) : \ X \longrightarrow Q_{-1}$ is a morphism between the following two complexes

$$\begin{tikzcd}
X = \cdots \arrow[r] & X^{i+1} \arrow[r, "\sigma^{i+1}"] \arrow[d, "\alpha_{i+1}"'] & X^i \arrow[r, "\sigma^i"] \arrow[d, "\alpha_i"'] & X^{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"'] & \cdots \\
Q_{-1} = \cdots \arrow[r] & Q^{i+1} \oplus Q^i \arrow[r, "\delta^{i+1}"] & Q^i \oplus Q^{i-1} \arrow[r, "\delta^i"] & Q^{i-1} \oplus Q^{i-2} \arrow[r] & \cdots
\end{tikzcd}$$

So we get an exact sequence of complexes

$$0 \longrightarrow X \xrightarrow{\alpha} Q_{-1} \longrightarrow K_{-1} \longrightarrow 0,$$

where $K_{-1}^i = \mathrm{Coker}\alpha_i, \forall i \in \mathbb{Z}$. It follows from Lemma 2.7 that each $K_{-1}^i \in \mathcal{D}_C\mathcal{P}(R)$. Thus the sequence of complexes

$$0 \longrightarrow \mathcal{H}\mathrm{om}(K_{-1}, H) \longrightarrow \mathcal{H}\mathrm{om}(Q_{-1}, H) \longrightarrow \mathcal{H}\mathrm{om}(X, H) \longrightarrow 0$$

is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$ by Lemma [25, Proposiiton 1.4]. Since $\mathcal{H}\mathrm{om}(X, H)$ is exact and $\mathcal{H}\mathrm{om}(Q_{-1}, H)$ is exact by Corollary 2.6, $\mathcal{H}\mathrm{om}(K_{-1}, H)$ is exact. Hence $\mathrm{Ext}^1(K_{-1}, H)$=0 by Lemma 2.3. This yields the following exact sequence

$$0 \longrightarrow \mathrm{Hom}(K_{-1}, H) \longrightarrow \mathrm{Hom}(Q_{-1}, H) \longrightarrow \mathrm{Hom}(X, H) \longrightarrow 0.$$

Note that K_{-1} has the same properties as X, we may use the same procedure to construct a $\mathrm{Hom}(-,\widetilde{\mathcal{F}_C(R)})$-exact exact sequence of complexes

$$0 \longrightarrow X \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots \qquad (\dagger)$$

with each $Q_i \in \widetilde{\mathcal{P}_C(R)}$ for any $i \in \mathbb{Z}$.

Take a projective resolution of X

$$\cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \longrightarrow 0. \qquad (\ddagger)$$

Set $K_j = \mathrm{Ker} f_j$, $j = 0, 1, 2, \ldots$. Then $K_j^i \in \mathcal{D}_C\mathcal{P}(R)$ by [25, Theorem 1.12] for any $j \geq 0$ and any $i \in \mathbb{Z}$. By the hypothesis and Lemma 2.3, the sequence

$$0 \longrightarrow \mathrm{Hom}(X, H) \longrightarrow \mathrm{Hom}(P_0, H) \longrightarrow \mathrm{Hom}(K_0, H) \longrightarrow 0$$

is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$. On the other hand, by Lemma [25, Proposition 1.4], the sequence of complexes

$$0 \longrightarrow \mathcal{H}\mathrm{om}(X, H) \longrightarrow \mathcal{H}\mathrm{om}(P_0, H) \longrightarrow \mathcal{H}\mathrm{om}(K_0, H) \longrightarrow 0$$

is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$ since each $X^i \in \mathcal{D}_C\mathcal{P}(R)$. Then $\mathcal{H}\mathrm{om}(K_0, H)$ is exact since both $\mathcal{H}\mathrm{om}(X, H)$ and $\mathcal{H}\mathrm{om}(P_0, H)$ are all exact. Continue this process one can prove that ($\ddagger$) is $\mathrm{Hom}(-,\widetilde{\mathcal{F}_C(R)})$-exact.

Now, assembling the sequence ($\dagger$) and ($\ddagger$) together, we get a $\mathrm{Hom}(-,\widetilde{\mathcal{F}_C(R)})$-exact exact sequence of complexes

$$\mathbf{E} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots$$

with all $P_i \in \widetilde{\mathcal{P}(R)}$ and all $Q_i \in \widetilde{\mathcal{P}_C(R)}$ such that $X \cong \mathrm{Im}(\ P_0 \longrightarrow Q_{-1})$. Therefore, X is a D_C-projective complex. $\square$

Corollary 2.9. *Projective complexes and C-projective complexes are D_C-projective complexes.*

Proof. It follows from Theorem 2.8, [25, Proposition 1.8] and Corollary 2.6. $\square$

Let $\mathcal{A}$ be an Abelian category. According to [16], a class $\mathcal{X}$ of objects of $\mathcal{A}$ is said to be projectively resolving if all projective objects of $\mathcal{A}$ are contained in $\mathcal{X}$ and for every short exact sequence $0 \longrightarrow X' \longrightarrow X \longrightarrow X'' \longrightarrow 0$ in $\mathcal{A}$, if $X'' \in \mathcal{X}$, then $X \in \mathcal{X}$ if and only if $X' \in \mathcal{X}$.

Corollary 2.10. *The class of D_C-projective complexes is projectively resolving.*

Proof. By Corollary 2.9, every projective complex is D_C-projective. Now consider an exact sequence in $\mathrm{Ch}(R)$

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

with Z D_C-projective. Then $Z^i \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(Z, H)$ is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$ by Theorem 2.8. So the sequence

$$0 \longrightarrow \mathcal{H}\mathrm{om}(Z, H) \longrightarrow \mathcal{H}\mathrm{om}(Y, H) \longrightarrow \mathcal{H}\mathrm{om}(X, H) \longrightarrow 0$$

is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$ by [25, Proposition 1.4]. Thus if X is a D_C-projective complex, then $X^i \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$ and $\mathcal{H}\mathrm{om}(X, H)$ is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$ by Theorem 2.8. Hence $\mathcal{H}\mathrm{om}(Y, H)$ is exact for any $H \in \widetilde{\mathcal{F}_C(R)}$, and by [25, Theorem 1.12], $Y^i \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$. Therefore Y is a D_C-projective complex by Theorem 2.8. The case Y is a D_C-projective complex can be proved similarly. $\square$

Corollary 2.11. *The class of D_C-projective complexes is closed under direct summands and direct sums.*

Proof. Suppose that X is a D_C-projective complex and $A \oplus B = X$. Then $A^i \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$ by Theorem 2.8 and [25, Theorem 1.12]. Let $H \in \widetilde{\mathcal{F}_C(R)}$. Then $\mathcal{H}\mathrm{om}(X, H)$ is exact by Theorem 2.8. Thus $\mathcal{H}\mathrm{om}(A, H)$ is exact since $\mathcal{H}\mathrm{om}(A, H) \oplus \mathcal{H}\mathrm{om}(B, H) \cong \mathcal{H}\mathrm{om}(A \oplus B, H)$. Hence A is a D_C-projective complex by Theorem 2.8.

Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a collection of D_C-projective complexes. Then $\bigoplus_{\lambda\in\Lambda} X_\lambda^i \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$ by Theorem 2.8 and [25, Proposition 1.11]. Let $H \in \widetilde{\mathcal{F}_C(R)}$. Then $\mathcal{H}\mathrm{om}(X_\lambda, H)$ is exact for any $\lambda \in \Lambda$. Since $\mathcal{H}\mathrm{om}(\bigoplus_{\lambda\in\Lambda} X_\lambda, H) \cong \prod_{\lambda\in\Lambda} \mathcal{H}\mathrm{om}(X_\lambda, H)$, $\mathcal{H}\mathrm{om}(\bigoplus_{\lambda\in\Lambda} X_\lambda, H)$ is exact. So $\bigoplus_{\lambda\in\Lambda} X_\lambda$ is a D_C-projective complex by Theorem 2.8. $\square$

Corollary 2.12. *Let* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be a short exact sequence of complexes. If X, Y are D_C-projective, then the following conditions are equivalent:*

(1) Z is a D_C-projective complex.

(2) $Z^i \in \mathcal{D}_C\mathcal{P}(R)$ for all $i \in \mathbb{Z}$.

(3) $\mathrm{Ext}^1(Z, H)$=0 for any $H \in \widetilde{\mathcal{F}_C(R)}$.

Proof. (1)$\Longrightarrow$(3) It follows from Lemma 2.2.

(3)$\Longrightarrow$(2) Let $i \in \mathbb{Z}$. Consider the exact sequence of R-modules

$$0 \longrightarrow X^i \longrightarrow Y^i \longrightarrow Z^i \longrightarrow 0.$$

By Theorem 2.8, $X^i, Y^i \in \mathcal{D}_C\mathcal{P}(R)$. It suffices to show that $\mathrm{Ext}^1(Z^i, F) = 0$ for any $F \in \mathcal{F}_C(R)$ by [25, Corollary 1.15].

Let $F \in \mathcal{F}_C(R)$. Then $\overline{F}[i] \in \widetilde{\mathcal{F}_C(R)}$. Thus $\mathrm{Ext}^1(Z, \overline{F}[i])$=0 by (3). Hence $\mathrm{Ext}^1(Z^i, F)$=0 since $\mathrm{Ext}^1(Z^i, F) \cong \mathrm{Ext}^1(Z, \overline{F}[i])$ by [13, Lemma 3.1].

(2)$\Longrightarrow$(1) Assume that $H \in \widetilde{\mathcal{F}_C(R)}$. Since each $Z^i \in \mathcal{D}_C\mathcal{P}(R)$, the sequence

$$0 \longrightarrow \mathcal{H}\mathrm{om}(Z, H) \longrightarrow \mathcal{H}\mathrm{om}(Y, H) \longrightarrow \mathcal{H}\mathrm{om}(X, H) \longrightarrow 0$$

is exact by [25, Proposition 1.4]. Then $\mathcal{H}\mathrm{om}(Z, H)$ is exact since $\mathcal{H}\mathrm{om}(X, H)$ and $\mathcal{H}\mathrm{om}(Y, H)$ are all exact. So Z is a D_C-projective complex by Theorem 2.8. $\square$

Corollary 2.13. *For any* $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$*-exact exact sequence*

$$\mathbf{E} = \cdots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} Q_{-1} \xrightarrow{f_{-1}} Q_{-2} \xrightarrow{f_{-2}} \cdots$$

with all $P_i \in \widetilde{\mathcal{P}(R)}$ *and all* $Q_i \in \widetilde{\mathcal{P}_C(R)}$*, each* $\mathrm{Coker} f_i$ *is a* D_C*-projective complex for any* $i \in \mathbb{Z}$*.*

Proof. Let $i \in \mathbb{Z}$, set $N_i = \mathrm{Coker} f_{i+1}$. Then N_0 is D_C-projective by the definition of D_C-projective complexes. Thus N_i is D_C-projective for any $i \geq 0$ by Corollaries 2.9 and 2.10. So it left to show that N_i is D_C-projective for any $i < 0$.

For any $m \in \mathbb{Z}$, we have a $\mathrm{Hom}(-, \mathcal{F}_C(R))$-exact exact sequence of R-modules

$$\mathbf{E}^m = \cdots \longrightarrow P_1^m \xrightarrow{f_1^m} P_0^m \xrightarrow{f_0^m} Q_{-1}^m \xrightarrow{f_{-1}^m} Q_{-2}^m \xrightarrow{f_{-2}^m} \cdots ,$$

where all $P_i^m \in \mathcal{P}(R)$ and all $Q_i^m \in \mathcal{P}_C(R)$. Then each $N_i^m = \mathrm{Coker} f_{i+1}^m \in \mathcal{D}_C\mathcal{P}(R)$ for any $i \in \mathbb{Z}$ by [25, Proposition 1.13]. Thus N_i is D_C-projective for any $i < 0$ inductively by Corollary 2.12. $\square$

The next two Lemmas play a crucial role in the rest of our discussion.

Lemma 2.14. *Let*

$$0 \longrightarrow A \longrightarrow G_1 \xrightarrow{f} G_0 \longrightarrow X \longrightarrow 0 \tag{2.1}$$

be an exact sequence in $\mathrm{Ch}(R)$ *with* G_0, G_1 D_C*-projective. Then*

(1) We have the following exact sequences

$$0 \longrightarrow A \longrightarrow Q \longrightarrow G \longrightarrow X \longrightarrow 0 \tag{2.2}$$

and

$$0 \longrightarrow A \longrightarrow W \longrightarrow P \longrightarrow X \longrightarrow 0 \tag{2.3}$$

with $Q \in \widetilde{\mathcal{P}_C(R)}$*,* $P \in \widetilde{\mathcal{P}(R)}$ *and* G, W D_C*-projective.*

(2) If the sequence (2.1) is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$*-exact, then so are (2.2) and (2.3).*

Proof. (1) Since G_1 is D_C-projective, there exists an exact sequence

$$0 \longrightarrow G_1 \longrightarrow Q \longrightarrow G_{-1} \longrightarrow 0$$

with $Q \in \widetilde{\mathcal{P}_C(R)}$ and G_{-1} D_C-projective by the definition of D_C-projective complexes and Corollary 2.13. Then we have the following pushout diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_1 \arrow[r] \arrow[d] & \mathrm{Im} f \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & Q \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
 & & G_{-1} \arrow[r, equal] \arrow[d] & G_{-1} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Consider the following pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathrm{Im} f \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & G_{-1} \arrow[r, equal] \arrow[d] & G_{-1} \arrow[d] & & \\
 & 0 & 0. & &
\end{tikzcd}$$

Connecting the middle rows in the above two diagrams, we get the exact sequence

$$0 \longrightarrow A \longrightarrow Q \longrightarrow G \longrightarrow X \longrightarrow 0.$$

Since G_0, G_{-1} are D_C-projective, then so is G by Corollary 2.10. Now the first desired exact sequence (2.2) follows.

Since G_0 is D_C-projective, there exists an exact sequence

$$0 \longrightarrow G^1 \longrightarrow P \longrightarrow G_0 \longrightarrow 0$$

with $P \in \widetilde{\mathcal{P}(R)}$ and G^1 D_C-projective. Then we have the following pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & G^1 \arrow[r, equal] \arrow[d] & G^1 \arrow[d] & & \\
0 \arrow[r] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Im} f \arrow[r] \arrow[d] & G_0 \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

And consider the following pullback diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & G^1 \arrow[r, equal] \arrow[d] & G^1 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & G_1 \arrow[r] \arrow[d] & \mathrm{Im} f \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since both G^1, G_1 are D_C-projective, so is W by Corollary 2.10. Connecting the middle rows in the above two diagrams, then we get the second desired exact sequence (2.3).

(2) Let $H \in \widetilde{\mathcal{F}_C(R)}$. Note that $\mathrm{Ext}^{i \geq 1}(X, H) = 0$ for any D_C-projective complex X by Lemma 2.2. If the exact sequence (2.1) is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact, then $\mathrm{Ext}^1(\mathrm{Im} f, H)$=0 and $\mathrm{Ext}^1(X, H) = \mathrm{Ext}^2(X, H) = 0$. So in the proof of (1), $\mathrm{Ext}^1(B, H)$=0 and $\mathrm{Ext}^1(N, H)$=0. Thus the exact sequences (2.2) and (2.3) are $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact. $\square$

Lemma 2.15. *Let $n \geq 1$ and*

$$0 \longrightarrow A \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow X \longrightarrow 0 \qquad (2.4)$$

an exact sequence in $\mathrm{Ch}(R)$ *with all* G_i D_C*-projective. Then*

(1) There exist exact sequences

$$0 \longrightarrow A \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow Y \longrightarrow 0 \tag{2.5}$$

and

$$0 \longrightarrow X \longrightarrow Y \longrightarrow U \longrightarrow 0$$

in $\mathrm{Ch}(R)$ *with all* $Q_i \in \widetilde{\mathcal{P}_C(R)}$ *and* U D_C*-projective.*

(2) There exist exact sequences

$$0 \longrightarrow B \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0 \tag{2.6}$$

and

$$0 \longrightarrow V \longrightarrow B \longrightarrow A \longrightarrow 0$$

in $\mathrm{Ch}(R)$ *with all* $P_i \in \widetilde{\mathcal{P}(R)}$ *and* V D_C*-projective.*

(3) If the exact sequence (2.4) is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$*-exact, then so are (2.5) and (2.6).*

Proof. (1) We proceed by induction on n.

When $n = 1$, we have an exact sequence $0 \longrightarrow A \longrightarrow G_0 \longrightarrow X \longrightarrow 0$ in $\mathrm{Ch}(R)$. Since G_0 is D_C-projective, we have a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence $0 \longrightarrow G_0 \longrightarrow Q_0 \longrightarrow U \longrightarrow 0$ with $Q_0 \in \widetilde{\mathcal{P}_C(R)}$ and U D_C-projective. Consider the following pushout diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & G_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & Q_0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \\
 & & U \arrow[r, equal] \arrow[d] & U \arrow[d] & \\
 & & 0 & 0. &
\end{tikzcd}$$

The middle row and the last column in the above diagram are the desired two exact sequences.

Now assume that $n \geq 2$ and we have an exact sequence

$$0 \longrightarrow A \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

in $\mathrm{Ch}(R)$ with all G_i D_C-projective. Put $K = \mathrm{Coker}(G_{n-1} \longrightarrow G_{n-2})$. By Lemma 2.14, we get an exact sequence

$$0 \longrightarrow A \longrightarrow Q_{n-1} \longrightarrow G'_{n-2} \longrightarrow K \longrightarrow 0 \tag{2.7}$$

in $\mathrm{Ch}(R)$ with $Q_{n-1} \in \widetilde{\mathcal{P}_C(R)}$ and G'_{n-2} D_C-projective. Set $A' = \mathrm{Im}(Q_{n-1} \longrightarrow G'_{n-2})$. Then we get an exact sequence

$$0 \longrightarrow A' \longrightarrow G'_{n-2} \longrightarrow G_{n-3} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

in $\mathrm{Ch}(R)$. Now we get the assertion by the induction hypothesis.

(2) The proof is dual to that of (1).

(3) If the exact sequence (2.4) is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact, then the middle rows in the above commutative diagram is also $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact. On the other hand, we can choose (2.7) to be $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact by Lemma 2.14. Then by the induction hypothesis, we can get (2.5) is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact. Dually, one gets another assertion. □

The following result means that an iteration of the procedure used to define the D_C-projective complexes yields exactly the D_C-projective complexes.

Theorem 2.16. *Let $X \in \mathrm{Ch}(R)$. Then X is D_C-projective if and only if there exists a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence of D_C-projective complexes*

$$\mathbf{G} = \cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

such that $X \cong$Cokerσ_1.

Proof. $\Longrightarrow$) It is trivial.

$\Longleftarrow$) Suppose that there exists a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence of D_C-projective complexes

$$\mathbf{G} = \cdots \longrightarrow G_1 \xrightarrow{\sigma_1} G_0 \xrightarrow{\sigma_0} G_{-1} \longrightarrow \cdots$$

such that $X \cong \mathrm{Im}\sigma_0$. Put $X_i = \mathrm{Im}\sigma_i$ for any $i \in \mathbb{Z}$. Then $X_0 = X$ and we have $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence in $\mathrm{Ch}(R)$

$$0 \longrightarrow X_{i+1} \longrightarrow G_i \longrightarrow X_i \longrightarrow 0$$

for all $i \in \mathbb{Z}$. We wish to construct an exact sequence of complexes satisfying the Definition 2.1.

Consider the short exact sequence

$$0 \longrightarrow X \longrightarrow G_{-1} \longrightarrow X_{-1} \longrightarrow 0.$$

By Lemma 2.15, there exist exact sequences

$$0 \longrightarrow X \longrightarrow Q_{-1} \longrightarrow Y_{-1} \longrightarrow 0$$

and

$$0 \longrightarrow X_{-1} \longrightarrow Y_{-1} \longrightarrow V_{-1} \longrightarrow 0$$

with $Q_{-1} \in \widetilde{\mathcal{P}_C(R)}$, V_{-1} D_C-projective and the former one is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact. Then by the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X_{-1} \arrow[r] \arrow[d] & Y_{-1} \arrow[r] \arrow[d] & V_{-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G_{-2} \arrow[r] \arrow[d] & U_{-1} \arrow[r] \arrow[d] & V_{-1} \arrow[r] & 0 \\
 & X_{-2} \arrow[r, equal] \arrow[d] & X_{-2} \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

we get an exact sequence

$$0 \longrightarrow Y_{-1} \longrightarrow U_{-1} \longrightarrow X_{-2} \longrightarrow 0.$$

By Corollary 2.10 and the exactness of the middle row in the above diagram, U_{-1} is D_C-projective. Since the first column in the above diagram is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact, $\mathrm{Ext}^1(X_{-2}, \widetilde{\mathcal{F}_C(R)}) = 0$. It yields that $0 \longrightarrow Y_{-1} \longrightarrow U_{-1} \longrightarrow X_{-2} \longrightarrow 0$ is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact. By Lemma 2.15, there exist exact sequences

$$0 \longrightarrow Y_{-1} \longrightarrow Q_{-2} \longrightarrow Y_{-2} \longrightarrow 0$$

and

$$0 \longrightarrow X_{-2} \longrightarrow Y_{-2} \longrightarrow V_{-2} \longrightarrow 0$$

with $Q_{-2} \in \widetilde{\mathcal{P}_C(R)}$, V_{-2} D_C-projective and the former one is $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact. Then by the above argument, we have a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence

$$0 \longrightarrow Y_{-2} \longrightarrow U_{-2} \longrightarrow X_{-3} \longrightarrow 0.$$

We proceed in this manner to get $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequences

$$0 \longrightarrow Y_{-i+1} \longrightarrow Q_{-i} \longrightarrow Y_{-i} \longrightarrow 0$$

with $Q_{-i} \in \widetilde{\mathcal{P}_C(R)}$ for $i = 1, 2, \ldots$ where $Y_0 = X$. Assembling these sequence together, we obtain a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence

$$0 \longrightarrow X \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots \qquad (*)$$

with all $Q_i \in \widetilde{\mathcal{P}_C(R)}$.

Dually, we can get a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0 \qquad (**)$$

with all $P_i \in \widetilde{\mathcal{P}(R)}$.

Finally, assembling the sequence $(*)$ and $(**)$, we get a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence

$$\mathbf{E} = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow Q_{-1} \longrightarrow Q_{-2} \longrightarrow \cdots,$$

with all $P_i \in \widetilde{\mathcal{P}(R)}$ and all $Q_i \in \widetilde{\mathcal{P}_C(R)}$ such that $X \cong \mathrm{Im}(\ P_0 \longrightarrow Q_{-1})$. So X is D_C-projective. $\square$

Corollary 2.17. *Let $X \in \mathrm{Ch}(R)$. Then X is D_C-projective if and only if there exists a $\mathrm{Hom}(-, \widetilde{\mathcal{F}_C(R)})$-exact exact sequence*

$$\cdots \longrightarrow W_1 \xrightarrow{\sigma_1} W_0 \xrightarrow{\sigma_0} W_{-1} \longrightarrow \cdots$$

in $\mathrm{Ch}(R)$ with all $W_i \in \widetilde{\mathcal{P}(R)} \cup \widetilde{\mathcal{P}_C(R)}$ such that $X \cong \mathrm{Coker}\sigma_1$.

Proof. Immediate from Corollary 2.9 and Theorem 2.16. $\square$

3. D_C-Projective Dimension of Complexes

Note that projective complexes are D_C-projective by Corollary 2.9, thus every complex admits a D_C-projective resolution. So we can define D_C-projective dimension of complexes as follows.

Definition 3.1. *Let $X \in \mathrm{Ch}(R)$. We will say that X has D_C-projective dimension less than or equal to n, denoted $D_C\text{-pd}(X) \leq n$, if there exists an exact sequence*

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

in $\mathrm{Ch}(R)$ with every G_i being D_C-projective. If no such finite sequence exists, define $D_C\text{-pd}(X) = \infty$, otherwise, if n is the least such integer, define $D_C\text{-pd}(X) = n$.

In this Section, we will give some criteria for computing $D_C\text{-pd}(X)$ of a complex X if $D_C\text{-pd}(X) < \infty$. For this purpose, we need the following result.

Lemma 3.2. *Let $X \in \mathrm{Ch}(R)$. Consider two exact sequences*

$$0 \longrightarrow H_n \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

and

$$0 \longrightarrow H'_n \longrightarrow G'_{n-1} \longrightarrow G'_{n-2} \longrightarrow \cdots \longrightarrow G'_0 \longrightarrow X \longrightarrow 0$$

with all G_i, G'_i are D_C-projective complexes. Then H_n is D_C-projective if and only if H'_n is D_C-projective.

Proof. Using Corollaries 2.10 and 2.11, the proof is similar to that of (i)$\Longrightarrow$(iii) in [2, Theorem 1.2.7]. □

A complex X is said to have $\mathcal{P}_C$-projective dimension less than or equal to n, denoted $\mathcal{P}_C$-pd$(X) \leq n$, if there is an exact sequence

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow X \longrightarrow 0$$

with each $Q_i \in \widetilde{\mathcal{P}_C(R)}$. If n is the least then we set $\mathcal{P}_C$-pd$(X) = n$, and if there is no such n then we set $\mathcal{P}_C$-pd$(X) = \infty$. The $\mathcal{F}_C$-flat dimension of X, denoted by $\mathcal{F}_C$-pd(X) can be defined similarly.

Theorem 3.3. *Let $X \in \mathrm{Ch}(R)$ and $n \geq 0$. Then the following are equivalent:*
(1) D_C-pd$(X) \leq n$.
(2) D_C-pd$(X) < \infty$ and $\mathrm{Ext}^m(X, H)$=0 for any $m > n$ and any $H \in \mathrm{Ch}(R)$ with $\mathcal{F}_C$-pd$(H) < \infty$.
(3) D_C-pd$(X) < \infty$ and $\mathrm{Ext}^m(X, H)$=0 for any $m > n$ and any $H \in \widetilde{\mathcal{F}_C(R)}$.
(4) For any exact sequence of complexes

$$\cdots \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

with all G_i D_C-projective, $K_n = \mathrm{Ker}(\, G_{n-1} \longrightarrow G_{n-2})$ is D_C-projective.
(5) For any integer t with $0 \leq t \leq n$, there is an exact sequence of complexes

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_{t+1} \longrightarrow G_t \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

such that G_t is D_C-projective, $Q_i \in \widetilde{\mathcal{P}_C(R)}$ for $i > t$ and $P_i \in \widetilde{\mathcal{P}(R)}$ for $i < t$.

Proof. (4)$\Longrightarrow$(1) and (2)$\Longrightarrow$(3) are trivial.
(1)$\Longrightarrow$(2) Since D_C-pd$(X) \leq n$, there is an exact sequence

$$0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

with G_i D_C-projective for all $0 \leq i \leq n$. Then by dimension shifting and Lemma 2.2, $\mathrm{Ext}^m(X, H) \cong \mathrm{Ext}^{m-n}(G_n, H)$=0 for any $m > n$ and any $H \in \mathrm{Ch}(R)$ with $\mathcal{F}_C$-pd$(H) < \infty$.

(3)$\Longrightarrow$(4) Let

$$\cdots \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

be an exact sequence in $\mathrm{Ch}(R)$ with all G_i D_C-projective. We will show that $K_n = \mathrm{Ker}(\ G_{n-1} \longrightarrow G_{n-2})$ is D_C-projective.

By (3), we assume that $D_C\text{-pd}(X) = m < \infty$. Then there is an exact sequence

$$0 \longrightarrow G'_m \longrightarrow G'_{m-1} \longrightarrow \cdots \longrightarrow G'_1 \longrightarrow G'_0 \longrightarrow X \longrightarrow 0$$

with all G'_i D_C-projective. If $m \leq n$, there is nothing to prove. Now we assume that $m > n$. Set $K'_i = \mathrm{Ker}(\ G'_{i-1} \longrightarrow G'_{i-2}\ )$ for $i = 1, 2, \ldots m$, where $G'_{-1} = X$ and $K'_m = G'_m$. By dimension shifting we have $\mathrm{Ext}^{i+1}(X, H) \cong \mathrm{Ext}^1(K'_i, H)$ for any $i = n, n+1, \ldots, m-1$ and any $H \in \widetilde{\mathcal{F}_C(R)}$. Thus K'_n is D_C-projective inductively by (3) and Corollary 2.12. Therefore K_n is D_C-projective by Lemma 3.2.

(1)$\Longrightarrow$(5) We proceed by induction on n.

If $n = 1$, then there exists an exact sequence $0 \longrightarrow D_1 \longrightarrow D_0 \longrightarrow X \longrightarrow 0$ with D_0, D_1 D_C-projective. By Lemma 2.14 with $A = 0$, we get the exact sequences $0 \longrightarrow Q_1 \longrightarrow G_0 \longrightarrow X \longrightarrow 0$ and $0 \longrightarrow G_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ with G_0, G_1 D_C-projective, $Q_1 \in \widetilde{\mathcal{P}_C(R)}$ and $P_0 \in \widetilde{\mathcal{P}(R)}$.

Next we suppose $n \geq 2$. Then there exists an exact sequence of complexes

$$0 \longrightarrow D_n \longrightarrow D_{n-1} \longrightarrow \cdots \longrightarrow D_0 \longrightarrow X \longrightarrow 0 \qquad (\star)$$

where all D_i are D_C-projective. Put $A = \mathrm{Ker}\ (D_1 \longrightarrow D_0)$. By applying Lemma 2.14 to the exact sequence

$$0 \longrightarrow A \longrightarrow D_1 \longrightarrow D_0 \longrightarrow X \longrightarrow 0,$$

one gets the exactness of

$$0 \longrightarrow A \longrightarrow D'_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with $P_0 \in \widetilde{\mathcal{P}(R)}$ and D'_1 D_C-projective. Hence we obtain the following exact sequence of complexes

$$0 \longrightarrow D_n \longrightarrow D_{n-1} \longrightarrow \cdots \longrightarrow D_2 \longrightarrow D'_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

Set $Y = \mathrm{Ker}\ (P_0 \longrightarrow X)$. Then $D_C\text{-pd}(Y) \leq n-1$. By the induction hypothesis, we can get an exact sequence

$$0 \longrightarrow Q_n \longrightarrow \cdots \longrightarrow Q_{t+1} \longrightarrow G_t \longrightarrow P_{t-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$

with G_t D_C-projective, all $P_i \in \widetilde{\mathcal{P}(R)}$ for $i < t$ and all $Q_i \in \widetilde{\mathcal{P}_C(R)}$ for $i > t$, where $1 \leq t \leq n$.

Now it remains to show (5) for the case $t = 0$. In the sequence ($\star$), set $B = \mathrm{Ker}\,(D_0 \longrightarrow X)$. One gets the exactness of

$$0 \longrightarrow D_n \longrightarrow D_{n-1} \longrightarrow \cdots \longrightarrow D_1 \longrightarrow B \longrightarrow 0.$$

By the induction hypothesis, there is an exact sequence

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow G_1' \longrightarrow B \longrightarrow 0,$$

with G_1' D_C-projective and all $Q_i \in \widetilde{\mathcal{P}_C(R)}$ with $2 \leq i \leq n$. Set $A = \mathrm{Coker}\,(Q_3 \longrightarrow Q_2)$. For the exact sequence

$$0 \longrightarrow A \longrightarrow G_1' \longrightarrow D_0 \longrightarrow X \longrightarrow 0,$$

by Lemma 2.14, we get an exact sequence

$$0 \longrightarrow A \longrightarrow Q_1 \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

with G_0 D_C-projective and $Q_1 \in \widetilde{\mathcal{P}_C(R)}$. Thus we obtain the desired exact sequence

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow G_0 \longrightarrow X \longrightarrow 0$$

with G_0 D_C-projective and all $Q_i \in \widetilde{\mathcal{P}_C(R)}$ for $1 \leq i \leq n$.

(5)$\Longrightarrow$(1) follows from Corollary 2.9. $\square$

Using an argument as in the proof of [2, Corollary 1.2.9], we get the following result by Lemma 2.10 and Theorem 3.3.

Corollary 3.4. *Let* $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ *be an exact sequence in* $\mathrm{Ch}(R)$. *Then the following hold:*

(1) For any $n \geq 0$, if D_C-$\mathrm{pd}(Z) \leq n$, then D_C-$\mathrm{pd}(X) \leq n$ if and only if D_C-$\mathrm{pd}(Y) \leq n$. Consequently, D_C-$\mathrm{pd}(X) \leq \max\{D_C\text{-}\mathrm{pd}(Y), D_C\text{-}\mathrm{pd}(Z)\}$ and D_C-$\mathrm{pd}(Y) \leq \max\{D_C\text{-}\mathrm{pd}(X), D_C\text{-}\mathrm{pd}(Z)\}$.

(2) If D_C-$\mathrm{pd}(X) > D_C$-$\mathrm{pd}(Z)$ or D_C-$\mathrm{pd}(Y) > D_C$-$\mathrm{pd}(Z)$, then D_C-$\mathrm{pd}(X) = D_C$-$\mathrm{pd}(Y)$.

(3) If D_C-$\mathrm{pd}(Z) > 0$ and Y is D_C-projective, then D_C-$\mathrm{pd}(X) = D_C$-$\mathrm{pd}(Z) - 1$.

In particular, if two complexes in the sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ have finite D_C-projective dimension, then so is the third.

Let $\mathcal{F}$ be a class of objects of an Abelian category $\mathcal{A}$ and A an object of $\mathcal{A}$. Following [10], we say that a morphism $f : F \longrightarrow A$ is a $\mathcal{F}$-precover if $F \in \mathcal{F}$ and $\mathrm{Hom}(F', F) \longrightarrow \mathrm{Hom}(F', A) \longrightarrow 0$ is exact for each $F' \in \mathcal{F}$. If such f is an epimorphism, then we call $f : F \longrightarrow A$ is an epic $\mathcal{F}$-precover of A. Recall that A is said to have a special $\mathcal{F}$-precover if there is an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$ with $F \in \mathcal{F}$ and $\mathrm{Ext}^1(\mathcal{F}, K) = 0$. It is clear

that A has an epic $\mathcal{F}$-precover if it has a special $\mathcal{F}$-precover. For more details about precovers, readers can refer to [10]. The following result shows that a complex of finite D_C-projective dimension can be approximated by a complex of finite C-projective dimension and can also be approximated by a D_C-projective complex.

Corollary 3.5. *Let $X \in \mathrm{Ch}(R)$ with D_C-pd$(X) = n < \infty$. Then*

(1) There exists an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow G \longrightarrow 0$ in $\mathrm{Ch}(R)$ with G D_C-projective and $\mathcal{P}_C$-pd$(Y) = n$.

(2) X admits a special D_C-projective precover $0 \longrightarrow K \longrightarrow G \longrightarrow X \longrightarrow 0$ with $\mathcal{P}_C$-pd$(K) = n - 1$ if $n > 0$ and $K = 0$ if $n = 0$.

Proof. (1) If X is D_C-projective then the result holds by Corollary 2.13. Now assume that D_C-pd$(X) = n > 0$. Then we use Lemma 2.15(1) with $A = 0$ to get an exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow G \longrightarrow 0$ with G D_C-projective and $\mathcal{P}_C$-pd$(Y) \leq n$. By Corollary 3.4(2), we have D_C-pd$(Y) = n$, and thus $\mathcal{P}_C$-pd$(Y) = n$.

(2) If $n = 0$, it is trivial. Now assume that $n > 0$. By Theorem 3.3, there exists an exact sequence

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow G \longrightarrow X \longrightarrow 0$$

with G D_C-projective and all $Q_i \in \widetilde{\mathcal{P}_C(R)}$ for $1 \leq i \leq n$. Put $K = \mathrm{Ker}(G \longrightarrow X)$. Then we have an exact sequence $0 \longrightarrow K \longrightarrow G \longrightarrow X \longrightarrow 0$ with G D_C-projective and $\mathcal{P}_C$-pd$(K) \leq n - 1$. It follows from Corollary 3.4(3) that D_C-pd$(K) = D_C$-pd$(X) - 1 = n - 1$, and so $\mathcal{P}_C$-pd$(K) = n - 1$. Also by Theorem 3.3, $\mathrm{Ext}^1(G', K) = 0$ for any D_C-projective complex G'. This completes the proof. □

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